

Weighted Estimates for the iterated Commutators of Multilinear Maximal and Fractional Type Operators

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Abstract

In this paper, the following iterated commutators $T_{*,\Pi b}$ of maximal operator for multilinear singular integral operators and $I_{\alpha,\Pi b}$ of multilinear fractional integral operator are introduced and studied

$$T_{*,\Pi b}(\vec{f})(x) = \sup_{\delta>0} \left| [b_1, [b_2, \dots [b_{m-1}, [b_m, T_\delta]_m]_{m-1} \dots]_1(\vec{f})(x) \right|,$$

$$I_{\alpha,\Pi b}(\vec{f})(x) = [b_1, [b_2, \dots [b_{m-1}, [b_m, I_\alpha]_m]_{m-1} \dots]_1(\vec{f})(x),$$

where T_δ are the smooth truncations of the multilinear singular integral operators and I_α is the multilinear fractional integral operator, $b_i \in BMO$ for $i = 1, \dots, m$ and $\vec{f} = (f_1, \dots, f_m)$.

Weighted strong and $L(\log L)$ type end-point estimates for the above iterated commutators associated with two class of multiple weights $A_{\vec{p}}$ and $A_{(\vec{p},q)}$ are obtained, respectively.

1 Introduction

The multilinear Calderón-Zygmund theory is a natural generalization of linear case. Many authors were interested in these topics ([6], [7], [5], [18], [15], [9], [19], [22], [4], [20], [25], [13] and [2]). So we first recall the definition and some results of multilinear Calderón-Zygmund operators as well as the corresponding multilinear maximal operators and fractional type operators.

Definition 1.1 (Multilinear Calderón-Zygmund operators) *Let T be a Multilinear operator initially defined on the m -fold product of Schwartz spaces and taking values in the space of tempered distributions,*

$$T : \mathcal{S}(\mathbb{R}^n) \times \dots \times \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}'(\mathbb{R}^n).$$

Following [6], we say that T is an m -linear Calderón-Zygmund operator if for some $1 \leq q_j < \infty$, it extends to a bounded multilinear operator from $L^{q_1} \times \dots \times L^{q_m}$ to L^q , where $\frac{1}{q} = \frac{1}{q_1} + \dots + \frac{1}{q_m}$,

2000 *Mathematics Subject Classification*: 42B20, 42B25

Key words and phrases: Multilinear Calderón-Zygmund operators, Maximal operators, Multilinear fractional type operators, Commutators, multiple weights $A_{\vec{p}}$ and $A_{(\vec{p},q)}$.

*The author was supported partly by NSFC (Grant No.10701010), NSFC (Key program Grant No.10931001), PCSIRT of China, Beijing Natural Science Foundation (Grant: 1102023).

and if there exists a function K , defined off the diagonal $x = y_1 = \cdots = y_m$ in $(\mathbb{R}^n)^{m+1}$, satisfying

$$T(f_1, \dots, f_m)(x) = \int_{(\mathbb{R}^n)^m} K(x, y_1, \dots, y_m) f_1(y_1) \cdots f_m(y_m) dy_1 \cdots dy_m$$

for all $x \notin \bigcap_{j=1}^m \text{supp } f_j$;

$$|K(y_0, y_1, \dots, y_m)| \leq \frac{A}{(\sum_{k,l=0}^m |y_k - y_l|)^{mn}}; \quad (1.1)$$

and

$$|K(y_0, \dots, y_j, \dots, y_m) - K(y_0, \dots, y'_j, \dots, y_m)| \leq \frac{A|y_j - y'_j|^\varepsilon}{(\sum_{k,l=0}^m |y_k - y_l|)^{mn+\varepsilon}}, \quad (1.2)$$

for some $\varepsilon > 0$ and all $0 \leq j \leq m$, whenever $|y_j - y'_j| \leq \frac{1}{2} \max_{0 \leq k \leq m} |y_j - y_k|$.

The maximal multilinear singular integral operator was defined by

$$T_*(\vec{f})(x) = \sup_{\delta > 0} |T_\delta(f_1, \dots, f_m)(x)|,$$

where T_δ are the smooth truncations of T given by

$$T_\delta(f_1, \dots, f_m)(x) = \int_{|x-y_1|^2 + \cdots + |x-y_m|^2 > \delta^2} K(x, y_1, \dots, y_m) f_1(y_1) \cdots f_m(y_m) d\vec{y}.$$

Here, $d\vec{y} = dy_1 \cdots dy_m$.

As is pointed in [17], $T_*(\vec{f})(x)$ is pointwise well-defined when $f_j \in L^{q_j}(\mathbb{R}^n)$ with $1 \leq q_j \leq \infty$.

The study of the multilinear singular integral operator and its maximal operator has a long history. For maximal multilinear operator T_* , one can see for example, [17], [14], [20] and [3] for more details. We list some results for T_* as follows:

Theorem A ([17]) Let $1 \leq q_i < \infty$, and q be such that $\frac{1}{q} = \frac{1}{q_1} + \cdots + \frac{1}{q_m}$, and $\omega \in A_{q_1} \cap \cdots \cap A_{q_m}$. Let T be an m -linear Calderón-Zygmund operator. Then there exists a constant $C_{q,n} < \infty$ so that for all $\vec{f} = (f_1, \dots, f_m)$ satisfying

$$\|T_*(\vec{f})\|_{L_\omega^q} \leq C_{n,q}(A + W) \prod_{i=1}^m \|f_i\|_{L_\omega^{q_i}},$$

where W is the norm of T in the mapping $T: L^1 \times \cdots \times L^1 \rightarrow L^{1/m, \infty}$.

Theorem B ([3]) Assume that $\frac{1}{p_1} + \cdots + \frac{1}{p_m} = \frac{1}{p}$ and $\vec{w} \in A_{\vec{p}}$, then

- (i) If $1 < p_1, \dots, p_m < \infty$, then T_* is bounded from $L^{p_1}(w_1) \times \cdots \times L^{p_m}(w_m)$ to $L^p(\vec{w})$;
- (ii) If $1 \leq p_1, \dots, p_m < \infty$, then T_* is bounded from $L^{p_1}(w_1) \times \cdots \times L^{p_m}(w_m)$ to $L^{p, \infty}(\vec{w})$.

Here, $A_{\vec{p}}$ is the multiple weights in the Definition 2.1 below. The boundedness of T_* on Hardy spaces and weighted Hardy spaces were obtained in [14] and [21].

Now, let's recall some definitions and background for the multilinear fractional type operators.

In 1992, Grafakos [12] first defined and studied the multilinear maximal function and multilinear fractional integral as follows

$$M_\alpha(\vec{f})(x) = \sup_{r>0} \frac{1}{r^{n-\alpha}} \int_{|y|<r} \left| \prod_{i=1}^m f_i(x - \theta_i y) \right| dy$$

and

$$I_\alpha(\vec{f})(x) = \int_{\mathbb{R}^n} \frac{1}{|y|^{n-\alpha}} \prod_{i=1}^m f_i(x - \theta_i y) dy,$$

where θ_i ($i = 1, \dots, m$) are fixed distinct and nonzero real numbers and $0 < \alpha < n$. We note that, if we simply take $m = 1$ and $\theta_i = 1$, then M_α and I_α are just the operators studied by Muckenhoupt and Wheeden in [23]. In 1999, Kenig and Stein [18] considered another more general type of multilinear fractional integral which was defined by

$$I_{\alpha,A}(\vec{f})(x) = \int_{(\mathbb{R}^n)^m} \frac{1}{|(y_1, \dots, y_m)|^{mn-\alpha}} \prod_{i=1}^m f_i(\ell_i(y_1, \dots, y_m, x)) dy_i,$$

where ℓ_i is a linear combination of y_j s and x depending on the matrix A . They showed that $I_{\alpha,A}$ was of strong type $(L^{p_1} \times \dots \times L^{p_m}, L^q)$ and weak type $(L^{p_1} \times \dots \times L^{p_m}, L^{q,\infty})$. When $\ell_i(y_1, \dots, y_m, x) = x - y_i$, we denote this multilinear fractional type operator by I_α .

For a long time, there is an open question ([16]) in the multilinear operators theory. That is, the existence of multiple weights theory for multilinear Calderón-Zygmund operators and multilinear fractional integral operators. This was established in [19], [22], [4] and the multiple weights $A_{\vec{p}}$ and $A_{(\vec{p},q)}$ were constructed (see the definitions in section 2 below).

In [19] and [4], the following commutators of T and I_α in the j -th entry were defined and studied, including weighted strong and weighted end-point $L(\log L)$ type estimates associated with $A_{\vec{p}}$ and $A_{(\vec{p},q)}$ weights, respectively.

Definition 1.2 (Commutators in the j -th entry) ([19], [4]) *Given a collection of locally integrable functions $\vec{b} = (b_1, \dots, b_m)$, we define the commutators of the m -linear Calderón-Zygmund operator T and fractional integral I_α to be*

$$[\vec{b}, T](\vec{f}) = T_{\vec{b}}(f_1, \dots, f_m) = \sum_{j=1}^m T_{\vec{b}}^j(\vec{f}), \quad I_{\vec{b},\alpha}(\vec{f})(x) = \sum_{i=1}^m I_{\vec{b},\alpha}^i(\vec{f})(x),$$

where each term is the commutator of b_j and T in the j -th entry of T , that is,

$$T_{\vec{b}}^j(\vec{f}) = b_j T(f_1, \dots, f_j, \dots, f_m) - T(f_1, \dots, b_j f_j, \dots, f_m).$$

Also

$$I_{\vec{b},\alpha}^i(\vec{f})(x) = b_i(x) I_\alpha(f_1, \dots, f_i, \dots, f_m)(x) - I_\alpha(f_1, \dots, b_i f_i, \dots, f_m)(x).$$

Recently, in [25], the following iterated commutators of multilinear Calderon-Zygmund operators and pointwise multiplication with functions in BMO are defined and studied in products of Lebesgue spaces, including strong type and weak end-point estimates with multiple $A_{\vec{p}}$ weights.

$$\begin{aligned} T_{\Pi b}(\vec{f})(x) &= [b_1, [b_2, \dots [b_{m-1}, [b_m, T]_m]_{m-1} \dots]_2]_1(\vec{f})(x) \\ &= \int_{(\mathbb{R}^n)^m} \prod_{j=1}^m (b_j(x) - b_j(y_j)) K(x, y_1, \dots, y_m) \prod_{i=1}^m f_i(y_i) d\vec{y}. \end{aligned} \quad (1.3)$$

Therefore, an open interesting question arises, can we establish the weighted strong and end-point estimates of the iterated commutators for the multilinear operator T_* and I_α ? We note that, there is no results for the commutators of multilinear operator T_* ($m \geq 2$), even for the commutators of T_* in the j -th entry.

In this article, we give a positive answer to the above question, we study iterated commutators of maximal multilinear singular integral operator and multilinear fractional integral operators defined by

$$\begin{aligned} T_{*, \Pi b}(\vec{f})(x) &= \sup_{\delta > 0} \left| [b_1, [b_2, \dots [b_{m-1}, [b_m, T_\delta]_m]_{m-1} \dots]_2]_1(\vec{f})(x) \right| \\ &= \sup_{\delta > 0} \left| \int_{|x-y_1|^2 + \dots + |x-y_m|^2 > \delta^2} \prod_{j=1}^m (b_j(x) - b_j(y_j)) K(x, y_1, \dots, y_m) \prod_{i=1}^m f_i(y_i) d\vec{y} \right| \end{aligned} \quad (1.4)$$

and

$$\begin{aligned} I_{\alpha, \Pi b}(\vec{f})(x) &= [b_1, [b_2, \dots [b_{m-1}, [b_m, I_\alpha]_m]_{m-1} \dots]_2]_1(\vec{f})(x) \\ &= \int_{(\mathbb{R}^n)^m} \frac{1}{|(x - y_1, \dots, x - y_m)|^{mn-\alpha}} \prod_{j=1}^m (b_j(x) - b_j(y_j)) \prod_{i=1}^m f_i(y_i) d\vec{y}. \end{aligned} \quad (1.5)$$

Remark 1.1 Note that, when $m = 1$ in (1.3), this definition coincides with the linear commutator $[b, T]f = bT(f) - T(bf)$ and $[b, I_\alpha]f = bI_\alpha(f) - I_\alpha(bf)$. One classical result given by Coifman, Rochberg and Weiss [8] is that $[b, T]$ is L^p bounded for $1 < p < \infty$ when $b \in BMO$. But $[b, T]$ fails to be an operator of weak type $(1, 1)$, a counterexample was given by C. Pérez and an alternative $L(\log L)$ type result was obtained in [24]. In 1982, Chanillo proved that the commutator of the fractional integral operator $[b, I_\alpha]$ is bounded from L^p into L^q ($p > 1, 1/q = 1/p - \alpha/n$) when $b \in BMO$. In 2002, Ding, Lu and Zhang [10] studied the continuity properties of fraction type operators. They showed that $[b, I_\alpha]$ fails to be an operator of weak type $(L^1, L^{n/(n-\alpha), \infty})$, counterexamples were given in [10], alternative $L(\log L)$ type estimates was obtained.

We state our results as follows.

Theorem 1.1 (Weighted strong bounds for $T_{*, \Pi b}$) Let $\vec{\omega} \in A_{\vec{p}}$, $\frac{1}{p} = \frac{1}{p_1} + \dots + \frac{1}{p_m}$ with $1 < p_j < \infty$, $j = 1, \dots, m$; and $\vec{b} \in (BMO)^m$. Then there is a constant $C > 0$ independent of \vec{b} and \vec{f} such that

$$\|T_{*, \Pi b}(\vec{f})\|_{L^p(\nu_{\vec{\omega}})} \leq C \prod_{j=1}^m \|b_j\|_{BMO} \prod_{i=1}^m \|f_i\|_{L^{p_i}(\omega_i)}, \quad (1.6)$$

where $\vec{b} = (b_1, \dots, b_m)$.

Theorem 1.2 (Weighted end-point estimate for $T_{*,\Pi b}$) Let $\vec{\omega} \in A_{(1,\dots,1)}$ and $\vec{b} \in (BMO)^m$. Then there exists a constant C depending on \vec{b} such that

$$\nu_{\vec{\omega}}\left(\left\{x \in \mathbb{R}^n : T_{*,\Pi b}(\vec{f})(x) > t^m\right\}\right) \leq C \left(\prod_{i=1}^m \int_{\mathbb{R}^n} \Phi^{(m)}\left(\frac{|f_i(y_i)|}{t}\right) \omega_i(y_i) dy_i\right)^{\frac{1}{m}}, \quad (1.7)$$

where $\Phi(t) = t(1 + \log^+ t)$ and $\Phi^{(m)} = \overbrace{\Phi \circ \dots \circ \Phi}^m$.

Remark 1.2 If $m = 1$, then weighted strong L^p and weighted end-point $L(\log L)$ estimates for commutators of the classical linear operator T_* were studied in [29].

As for $I_{\alpha,\Pi b}$, we get

Theorem 1.3 (Weighted strong bounds for $I_{\alpha,\Pi b}$) Let $0 < \alpha < mn$, $1 < p_1, \dots, p_m < \infty$, $\frac{1}{p} = \frac{1}{p_1} + \dots + \frac{1}{p_m}$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. For $r > 1$ with $0 < r\alpha < mn$, if $\vec{\omega}^r \in A_{(\frac{p}{r}, \dots, \frac{p}{r})}$, $\nu_{\vec{\omega}^q} \in A_\infty$ and $\vec{b} \in (BMO)^m$, there is a constant $C > 0$ independent of \vec{b} such that

$$\|I_{\alpha,\Pi b}(\vec{f})\|_{L^q(\nu_{\vec{\omega}^q})} \leq C \prod_{j=1}^m \|b_j\|_{BMO} \prod_{i=1}^m \|f_i\|_{L^{p_i}(\omega_i^{p_i})}. \quad (1.8)$$

Theorem 1.4 (Weighted end-point estimate for $I_{\alpha,\Pi b}$) Let $0 < \alpha < mn$, $\vec{\omega} \in A_{((1,\dots,1), \frac{n}{mn-\alpha})}$ and $\vec{b} \in (BMO)^m$. Then there exists a constant C depending on \vec{b} , such that

$$\begin{aligned} & \nu_{\vec{\omega}}^{\frac{n}{mn-\alpha}} \left(\left\{ x \in \mathbb{R}^n : I_{\alpha,\Pi b}(\vec{f})(x) > t^{\frac{mn-\alpha}{n}} \right\} \right) \\ & \leq C \left\{ \left[1 + \frac{\alpha}{mn} \log^+ \left(\prod_{i=1}^m \int_{\mathbb{R}^n} \Phi^{(m)}\left(\frac{|f_i(y_i)|}{t}\right) dy_i \right) \right]^m \prod_{j=1}^m \int_{\mathbb{R}^n} \Phi^{(m)}\left(\frac{|f_j(y_j)|}{t}\right) \omega_j(y_j) dy_j \right\}^{\frac{n}{mn-\alpha}}. \end{aligned} \quad (1.9)$$

Moreover, if each $0 < \alpha_j < n$, we obtain

$$\begin{aligned} & \nu_{\vec{\omega}}^{\frac{n}{mn-\alpha}} \left(\left\{ x \in \mathbb{R}^n : I_{\alpha,\Pi b}(\vec{f})(x) > t^{\frac{mn-\alpha}{n}} \right\} \right) \\ & \leq C \left\{ \prod_{j=1}^m \left[1 + \frac{\alpha_j}{n} \log^+ \left(\prod_{i=1}^m \int_{\mathbb{R}^n} \Phi^{(m)}\left(\frac{|f_i(y_i)|}{t}\right) dy_i \right) \right] \int_{\mathbb{R}^n} \Phi^{(m)}\left(\frac{|f_j(y_j)|}{t}\right) \omega_j(y_j) dy_j \right\}^{\frac{n}{mn-\alpha}}, \end{aligned} \quad (1.10)$$

where $\Phi(t)$ and $\Phi^{(m)}$ are the same as in Theorem 1.2.

As a corollary of Theorem 1.3 and Theorem 1.4, we can obtain similar results for the commutators of the multilinear fractional maximal operator. Let's first give its definition. Suppose each f_i ($i = 1, \dots, m$) is locally integrable on \mathbb{R}^n . Then for any $x \in \mathbb{R}^n$, we define the multilinear fractional maximal operator and its commutators by

$$\mathcal{M}_\alpha(\vec{f})(x) = \sup_Q |Q|^{\frac{\alpha}{n}} \prod_{i=1}^m \frac{1}{|Q|} \int_Q |f_i(y_i)| dy_i,$$

and

$$\mathcal{M}_{\alpha, \Pi b}(\vec{f})(x) = \sup_Q |Q|^{\frac{\alpha}{n}} \prod_{i=1}^m \frac{1}{|Q|} \int_Q |b_i(x) - b_i(y_i)| |f_i(y_i)| dy_i,$$

where the supremum is taken over all cubes Q containing x in \mathbb{R}^n with the sides parallel to the axes.

Corollary 1.1 *Let $\alpha, b_i, \vec{\omega}, p_i, q$ be the same as in Theorem 1.3-1.4, then Theorem 1.3-1.4 still hold for $\mathcal{M}_{\alpha, \Pi b}$.*

The article is organized as follows. In section 2, we prepare some definitions and lemmas. Some propositions will be listed and proved in section 3, including the main Proposition 3.1. Then, we give the proof of Theorem 1.1-1.3. Section 4 will be devoted to the study of the end-point $L(\log L)$ type estimates for the iterated commutators of multilinear fractional type operators.

2 Definitions and some lemmas

Let us recall the definitions of $A_{\vec{p}}$ and $A_{(\vec{p}, q)}$ weights.

For m -exponents p_1, \dots, p_m , we will often write p for the number given by $\frac{1}{p} = \frac{1}{p_1} + \dots + \frac{1}{p_m}$, and \vec{p} for the vector $\vec{p} = (p_1, \dots, p_m)$.

Definition 2.1 (Multiple $A_{\vec{p}}$ weights) ([19]) *Let $1 \leq p_1, \dots, p_m < \infty$. Given $\vec{\omega} = (\omega_1, \dots, \omega_m)$, set*

$$\nu_{\vec{\omega}} = \prod_{j=1}^m \omega_j^{p/p_j}.$$

We say that $\vec{\omega}$ satisfies the $A_{\vec{p}}$ condition if

$$\sup_Q \left(\frac{1}{|Q|} \int_Q \prod_{i=1}^m \omega_i^{\frac{p}{p_i}} \right)^{\frac{1}{p}} \prod_{i=1}^m \left(\frac{1}{|Q|} \int_Q \omega_i^{1-p'_i} \right)^{\frac{1}{p'_i}} < \infty. \quad (2.1)$$

When $p_j = 1$, $\left(\frac{1}{|Q|} \int_Q \omega_i^{1-p'_i} \right)^{\frac{1}{p'_i}}$ is understood as $(\inf_Q \omega_i)^{-1}$.

Definition 2.2 (Multiple $A_{(\vec{p}, q)}$ weights) ([4], [22]) *Let $1 \leq p_1, \dots, p_m < \infty$, $\frac{1}{p} = \frac{1}{p_1} + \dots + \frac{1}{p_m}$, and $q > 0$. Suppose that $\vec{\omega} = (\omega_1, \dots, \omega_m)$ and each ω_i ($i = 1, \dots, m$) is a nonnegative function on \mathbb{R}^n . We say that $\vec{\omega} \in A_{(\vec{p}, q)}$ if it satisfies*

$$\sup_Q \left(\frac{1}{|Q|} \int_Q \nu_{\vec{\omega}}^q \right)^{\frac{1}{q}} \prod_{i=1}^m \left(\frac{1}{|Q|} \int_Q \omega_i^{-p'_i} \right)^{\frac{1}{p'_i}} < \infty, \quad (2.2)$$

where $\nu_{\vec{\omega}} = \prod_{i=1}^m \omega_i$. If $p_i = 1$, $\left(\frac{1}{|Q|} \int_Q \omega_i^{-p'_i} \right)^{\frac{1}{p'_i}}$ is understood as $(\inf_Q \omega_i)^{-1}$.

Remark 2.1 In particular, when $m = 1$, we note that $A_{\vec{p}}$ will be degenerated to the classical A_p weights. Moreover, if $m=1$ and $p_i = 1$, then this class of weights coincide with the classical A_1 weights. Also, when $m = 1$, we note that $A_{(\vec{p},q)}$ will be degenerated to the classical $A_{(p,q)}$ weights, where the latter was defined in 1974 by B. Muckenhoupt and R. Wheeden [23]. We will refer to (1.4) and (1.5) as the multilinear $A_{\vec{p}}$ condition and $A_{(\vec{p},q)}$ condition.

We need the following $L(\log L)$ type multilinear maximal fractional operators

Definition 2.3 For any $\vec{f} = (f_1, \dots, f_m)$ and $0 < \alpha < mn$ with $\sum_{i=1}^m \alpha_i = \alpha$, two multilinear fractional $L(\log L)$ type maximal operators are defined as

$$\mathcal{M}_{L(\log L), \alpha}^j(\vec{f})(x) = \sup_{Q \ni x} |Q|^{\frac{\alpha}{n}} \|f_j\|_{L(\log L), Q} \prod_{i \neq j} \frac{1}{|Q|} \int_Q |f_i|$$

and

$$\mathcal{M}_{L(\log L), \alpha}(\vec{f})(x) = \sup_{Q \ni x} |Q|^{\frac{\alpha}{n}} \prod_{i=1}^m \|f_i\|_{L(\log L), Q},$$

respectively. If $\alpha = 0$, for simply, we denote $\mathcal{M}_{L(\log L), 0} = \mathcal{M}_{L(\log L)}$ and $\mathcal{M}_{L(\log L), 0}^j = \mathcal{M}_{L(\log L)}^j$

We prepare some lemmas which will be used later. The following Hölder's inequality on Orlicz spaces can be seen in [27, p. 58].

Lemma 2.1 (Generalized Hölder's inequality) ([27]) Let $\phi(t) = t(1 + \log^+ t)$ and $\psi(t) = e^t - 1$ and suppose that

$$\begin{aligned} \|f\|_{\phi} &\triangleq \inf \left\{ \lambda > 0 : \int_{\mathbb{R}^n} \phi\left(\frac{|f(x)|}{\lambda}\right) d\mu \leq 1 \right\} < \infty \\ \|g\|_{\psi} &\triangleq \inf \left\{ \lambda > 0 : \int_{\mathbb{R}^n} \psi\left(\frac{|g(x)|}{\lambda}\right) d\mu \leq 1 \right\} < \infty \end{aligned}$$

with respect to some measure μ , then for any cube Q

$$\frac{1}{|Q|} \int_Q |fg| \leq 2 \|f\|_{L(\log L), Q} \|g\|_{\exp L, Q}. \quad (2.3)$$

Some other inequalities are also necessary.

Lemma 2.2 ([4]) Suppose that $r > 1$ and $b \in BMO$, then for any f satisfying the condition of generalized Hölder's inequality there is a $C > 0$ independent of \vec{f} and b such that

$$\frac{1}{|Q|} \int_Q |f| \leq C \|f\|_{L(\log L), Q}; \quad (2.4)$$

$$\|f\|_{L(\log L), Q} \leq C \left(\frac{1}{|Q|} \int_Q |f|^r \right)^{\frac{1}{r}}; \quad (2.5)$$

$$\frac{1}{|Q|} \int_Q |(b - b_Q)f| \leq C \|b\|_{BMO} \|f\|_{L(\log L), Q}; \quad (2.6)$$

$$\left(\sup_Q \frac{1}{|Q|} \int_Q |b - b_Q|^{r-1} \right)^{\frac{1}{r-1}} \leq C \|b\|_{BMO}. \quad (2.7)$$

We need Kolmogorov's inequalities in the following lemma, which are necessary tools for some estimates.

Lemma 2.3 (Kolmogorov's inequality) ([19], [11, p. 485])

(a) Suppose $0 < p < q < \infty$, then

$$\|f\|_{L^p(Q, \frac{dx}{|Q|})} \leq C \|f\|_{L^{q,\infty}(Q, \frac{dx}{|Q|})}; \quad (2.8)$$

(b) Suppose that $0 < \alpha < n$ and $p, q > 0$ satisfying $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. Then for any measurable function f and cube Q ,

$$\left(\int_Q |f|^p \right)^{\frac{1}{p}} \leq \left(\frac{q}{q-p} \right)^{\frac{1}{p}} |Q|^{\frac{\alpha}{n}} \|f\|_{L^{q,\infty}(Q)}. \quad (2.9)$$

To prove Theorem 1.4, we also need the following known results,

Lemma 2.4 (Weighted estimates for \mathcal{M}_α and I_α) ([22], [4]) Let $0 < \alpha < mn$, $1 \leq p_1, \dots, p_m < \infty$, $\frac{1}{p} = \frac{1}{p_1} + \dots + \frac{1}{p_m}$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. Then for $\vec{\omega} \in A_{(\vec{p},q)}$ there is a constant $C > 0$ independent of \vec{f} such that

$$\|\mathcal{M}_\alpha(\vec{f})\|_{L^{q,\infty}(\nu_{\vec{\omega}}^q)} \leq C \prod_{i=1}^m \|f_i\|_{L^{p_i}(\omega_i^{p_i})}; \quad (2.10)$$

$$\|I_\alpha(\vec{f})\|_{L^{q,\infty}(\nu_{\vec{\omega}}^q)} \leq C \prod_{i=1}^m \|f_i\|_{L^{p_i}(\omega_i^{p_i})}. \quad (2.11)$$

3 Proof of Theorem 1.1-1.3

To begin with, we prepare one proposition which plays important role in the proof of our theorems. The basic idea is to control the iterated commutators of T_* by another two operators.

Let $u, v \in C^\infty([0, \infty))$ such that $|u'(t)| \leq Ct^{-1}$, $|v'(t)| \leq Ct^{-1}$ and satisfy

$$\chi_{[2,\infty)}(t) \leq u(t) \leq \chi_{[1,\infty)}(t), \quad \chi_{[1,2]}(t) \leq v(t) \leq \chi_{[1/2,3]}(t).$$

We define the maximal operators

$$U^*(\vec{f})(x) = \sup_{\eta > 0} \left| \int_{(\mathbb{R}^n)^m} K(x, y_1, \dots, y_m) u(\sqrt{|x - y_1| + \dots + |x - y_m|}/\eta) \prod_{i=1}^m f_i(y_i) d\vec{y} \right|,$$

$$V^*(\vec{f})(x) = \sup_{\eta > 0} \left| \int_{(\mathbb{R}^n)^m} K(x, y_1, \dots, y_m) v(\sqrt{|x - y_1| + \dots + |x - y_m|}/\eta) \prod_{i=1}^m f_i(y_i) d\vec{y} \right|.$$

For simplicity, we denote $K_{u,\eta}(x, y_1, \dots, y_m) = K(x, y_1, \dots, y_m)u(\sqrt{|x - y_1| + \dots + |x - y_m|}/\eta)$, $K_{v,\eta}(x, y_1, \dots, y_m) = K(x, y_1, \dots, y_m)v(\sqrt{|x - y_1| + \dots + |x - y_m|}/\eta)$ and

$$U_\eta(\vec{f}) = \int_{(\mathbb{R}^n)^m} K_{u,\eta}(x, y_1, \dots, y_m) \prod_{i=1}^m f_i(y_i) d\vec{y}$$

and

$$V_\eta(\vec{f}) = \int_{(\mathbb{R}^n)^m} K_{v,\eta}(x, y_1, \dots, y_m) \prod_{i=1}^m f_i(y_i) d\vec{y}.$$

It is easy to see that $T_*(\vec{f}) \leq U^*(\vec{f})(x) + V^*(\vec{f})(x)$. Moreover, $T_{*,\Pi b}(\vec{f}) \leq U_{\Pi b}^*(\vec{f})(x) + V_{\Pi b}^*(\vec{f})(x)$, where

$$\begin{aligned} U_{\Pi b}^*(\vec{f})(x) &= \sup_{\eta>0} \left| [b_1, [b_2, \dots [b_{m-1}, [b_m, U_\eta]_m]_{m-1} \dots]_2]_1(\vec{f})(x) \right| \\ &= \sup_{\eta>0} \left| \int_{(\mathbb{R}^n)^m} K_{u,\eta}(x, y_1, \dots, y_m) \prod_{j=1}^m (b_j(x) - b_j(y_j)) \prod_{i=1}^m f_i(y_i) d\vec{y} \right| \end{aligned}$$

and

$$\begin{aligned} V_{\Pi b}^*(\vec{f})(x) &= \sup_{\eta>0} \left| [b_1, [b_2, \dots [b_{m-1}, [b_m, V_\eta]_m]_{m-1} \dots]_2]_1(\vec{f})(x) \right| \\ &= \sup_{\eta>0} \left| \int_{(\mathbb{R}^n)^m} K_{v,\eta}(x, y_1, \dots, y_m) \prod_{j=1}^m (b_j(x) - b_j(y_j)) \prod_{i=1}^m f_i(y_i) d\vec{y} \right|. \end{aligned}$$

Following [25], for positive integers m and j with $1 \leq j \leq m$, we denote by C_j^m the family of all finite subsets $\sigma = \{\sigma(1), \dots, \sigma(j)\}$ of $\{1, \dots, m\}$ of j different elements, where we always take $\sigma(k) < \sigma(j)$ if $k < j$. For any $\sigma \in C_j^m$, we associated the complementary sequence $\sigma' \in C_j^{m-j}$ given by $\sigma' = \{1, \dots, m\} \setminus \sigma$ with the convention $C_0^m = \emptyset$. Given an m -tuple of functions b and $\sigma \in C_j^m$, we also use the notation b_σ for the j -tuple obtained from b given by $(b_{\sigma(1)}, \dots, b_{\sigma(j)})$.

Similarly to the above definition for $U_{\Pi b}^*(\vec{f})(x)$ and $V_{\Pi b}^*(\vec{f})(x)$, $\sigma \in C_j^m$, and $b_\sigma = (b_{\sigma(1)}, \dots, b_{\sigma(j)})$ in BMO^j , the iterated commutator

$$\begin{aligned} U_{\Pi b_\sigma}^*(\vec{f})(x) &= \sup_{\eta>0} \left| \int_{(\mathbb{R}^n)^m} K_{u,\eta}(x, y_1, \dots, y_m) \prod_{i=1}^j (b_{\sigma(i)}(x) - b_{\sigma(i)}(y_{\sigma(i)})) \prod_{i=1}^m f_i(y_i) d\vec{y} \right|; \\ V_{\Pi b_\sigma}^*(\vec{f})(x) &= \sup_{\eta>0} \left| \int_{(\mathbb{R}^n)^m} K_{v,\eta}(x, y_1, \dots, y_m) \prod_{i=1}^j (b_{\sigma(i)}(x) - b_{\sigma(i)}(y_{\sigma(i)})) \prod_{i=1}^m f_i(y_i) d\vec{y} \right|; \\ I_{\alpha, \Pi b_\sigma}(\vec{f})(x) &= \int_{(\mathbb{R}^n)^m} \frac{1}{|(x - y_1, \dots, x - y_m)|^{mn-\alpha}} \prod_{i=1}^j (b_{\sigma(i)}(x) - b_{\sigma(i)}(y_{\sigma(i)})) \prod_{i=1}^m f_i(y_i) d\vec{y}. \end{aligned}$$

While $\sigma = \{j\}$, $U_{\Pi b_\sigma}^*(\vec{f}) = U_{b_j}^*(\vec{f})$, $V_{\Pi b_\sigma}^*(\vec{f}) = V_{b_j}^*(\vec{f})$ and $I_{\alpha, \Pi b_\sigma}(\vec{f}) = I_{b_j, \alpha}^j(\vec{f})$. If $\sigma = \{1, \dots, m\}$, then $U_{\Pi b_\sigma}^*(\vec{f}) = U_{\Pi b}^*(\vec{f})$, $V_{\Pi b_\sigma}^*(\vec{f}) = V_{\Pi b}^*(\vec{f})$ and $I_{\alpha, \Pi b_\sigma}(\vec{f}) = I_{\alpha, \Pi b}^j(\vec{f})$.

Proposition 3.1 (Pointwise control of $M_\delta^\sharp(U_{\Pi b}^*(\vec{f})), M_\delta^\sharp(V_{\Pi b}^*(\vec{f})), M_\delta^\sharp(I_{\alpha, \Pi b}(\vec{f}))$) Let $0 < \delta < \varepsilon$, $0 < \delta < \frac{1}{m}$ and $0 < \alpha < mn$. Then there is a constant $C > 0$ depending on δ and ε such that

$$\begin{aligned} M_\delta^\sharp(U_{\Pi b}^*(\vec{f}))(x) &\leq C \prod_{j=1}^m \|b_j\|_{BMO} (\mathcal{M}_{L(\log L)}(\vec{f})(x) + M_\varepsilon(U^*(\vec{f}))(x)) \\ &\quad + C \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} \prod_{i=1}^j \|b_{\sigma(i)}\|_{BMO} M_\varepsilon(U_{\Pi b_{\sigma'}}^*(\vec{f}))(x), \end{aligned} \quad (3.1)$$

$$\begin{aligned} M_\delta^\sharp(V_{\Pi b}^*(\vec{f}))(x) &\leq C \prod_{j=1}^m \|b_j\|_{BMO} (\mathcal{M}_{L(\log L)}(\vec{f})(x) + M_\varepsilon(V^*(\vec{f}))(x)) \\ &\quad + C \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} \prod_{i=1}^j \|b_{\sigma(i)}\|_{BMO} M_\varepsilon(V_{\Pi b_{\sigma'}}^*(\vec{f}))(x), \end{aligned} \quad (3.2)$$

$$\begin{aligned} M_\delta^\sharp(I_{\alpha, \Pi b}(\vec{f}))(x) &\leq C \prod_{j=1}^m \|b_j\|_{BMO} (\mathcal{M}_{L(\log L), \alpha}(\vec{f})(x) + M_\varepsilon(I_\alpha(\vec{f}))(x)) \\ &\quad + C \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} \prod_{i=1}^j \|b_{\sigma(i)}\|_{BMO} M_\varepsilon(I_{\alpha, \Pi b_{\sigma'}}(\vec{f}))(x). \end{aligned} \quad (3.3)$$

(3.3) still hold for $\delta = 1/m$.

Proof of Proposition 3.1.

We only give the proof for $U_{\Pi b}^*(\vec{f})$ and $I_{\alpha, \Pi b}(\vec{f})$, since the proof for $V_{\Pi b}^*(\vec{f})$ is almost the same as $U_{\Pi b}^*(\vec{f})$.

For simplicity, we only prove for the case $m = 2$, since there is no essential difference for the general case. Fix $b_1, b_2 \in BMO$ and denote any constants by ρ_1, ρ_2 . We split $U_{\Pi b}^*(\vec{f})(x)$ in the following way,

$$\begin{aligned} U_{\Pi b}^*(\vec{f})(x) &= \sup_{\eta > 0} |(b_1(x) - \rho_1)(b_2(x) - \rho_2)U_\eta(\vec{f})(x) - (b_1(x) - \rho_1)U_\eta(f_1, (b_2 - \rho_2)f_2)(x) \\ &\quad - (b_2(x) - \rho_2)U_\eta((b_1 - \rho_1)f_1, f_2)(x) + U_\eta((b_1 - \rho_1)f_1, (b_2 - \rho_2)f_2)(x)| \\ &= \sup_{\eta > 0} |-(b_1(x) - \rho_1)(b_2(x) - \rho_2)U_\eta(\vec{f})(x) + (b_1(x) - \rho_1)U_{\eta, b_2 - \rho_2}^2(f_1, f_2)(x) \\ &\quad + (b_2(x) - \rho_2)U_{\eta, b_1 - \rho_1}^1(f_1, f_2)(x) + U_\eta((b_1 - \rho_1)f_1, (b_2 - \rho_2)f_2)(x)|. \end{aligned}$$

Here we denote $U_{\eta, b_1 - \rho_1}^1(f_1, f_2)(x) = U_\eta((b_1 - \rho_1)f_1, f_2)(x)$ and $U_{\eta, b_2 - \rho_2}^2(f_1, f_2)(x) = U_\eta(f_1, (b_2 - \rho_2)f_2)(x)$, similar notation will be used in the rest of this paper.

Fix $x_0 \in \mathbb{R}^n$ and let Q be a cube centered at x_0 . Since $0 < \delta < \frac{1}{m}$, Let $c = \sup_\eta |\sum_{j=1}^3 c_j|$, then we have

$$\left(\frac{1}{|Q|} \int_Q |U_{\Pi b}^*(\vec{f})(z)|^\delta - |c|^\delta dz \right)^{\frac{1}{\delta}} \leq C(T_1 + T_2 + T_3 + T_4),$$

where

$$T_1 = \left(\frac{1}{|Q|} \int_Q |(b_1(z) - \rho_1)(b_2(z) - \rho_2)|^\delta U^*(\vec{f})(z)^\delta dz \right)^{\frac{1}{\delta}},$$

$$T_2 = \left(\frac{1}{|Q|} \int_Q \sup_{\eta>0} |(b_1(z) - \rho_1)[U_{\eta, b_2 - \rho_2}^2(f_1, f_2)(z)]|^\delta dz \right)^{\frac{1}{\delta}}.$$

$$T_3 = \left(\frac{1}{|Q|} \int_Q \sup_{\eta>0} |(b_2(z) - \rho_2)[U_{\eta, b_1 - \rho_1}^1(f_1, f_2)(z)]|^\delta dz \right)^{\frac{1}{\delta}}$$

and

$$T_4 = \left(\frac{1}{|Q|} \int_Q \sup_{\eta>0} |U_\eta((b_1 - \rho_1)f_1, (b_2 - \rho_2)f_2)(z) - \sum_{j=1}^3 c_j|^\delta dz \right)^{\frac{1}{\delta}}$$

Let $\rho_j = (b_j)_{3Q}$ be the average of b_j on $3Q$ for $j = 1, 2$.

For any $1 < r_1, r_2, r_3 < \infty$ with $\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} = 1$ and $r_3 < \frac{\varepsilon}{\delta}$, T_1 can be estimated by using the Hölder's inequality and (2.7).

$$\begin{aligned} T_1 &\leq C \left(\frac{1}{|Q|} \int_Q |b_1(z) - \rho_1|^{\delta r_1} dz \right)^{\frac{1}{\delta r_1}} \left(\frac{1}{|Q|} \int_Q |b_2(z) - \rho_2|^{\delta r_2} dz \right)^{\frac{1}{\delta r_2}} \\ &\quad \times \left(\frac{1}{|Q|} \int_Q |U^*(\vec{f})(z)|^{\delta r_3} dz \right)^{\frac{1}{\delta r_3}} \\ &\leq C \prod_{j=1}^2 \|b_j\|_{BMO} M_\varepsilon(U^*(\vec{f}))(x_0). \end{aligned}$$

Since T_2 and T_3 are symmetric we only estimate T_2 . Let $1 < t_1, t_2 < \infty$ with $1 = 1/t_1 + 1/t_2$ and $t_2 < \frac{\varepsilon}{\delta}$, then T_1 can be estimated by using the Hölder's inequality and Jensen's inequalities,

$$\begin{aligned} T_2 &\leq C \left(\frac{1}{|Q|} \int_Q |b_1(z) - \rho_1|^{\delta t_1} dz \right)^{\frac{1}{\delta t_1}} \left(\frac{1}{|Q|} \int_Q \sup_{\eta>0} |U_{\eta, b_2 - \rho_2}^2(f_1, f_2)(z)|^{\delta t_2} dz \right)^{\frac{1}{\delta t_2}} \\ &\leq C \|b_1\|_{BMO} M_\varepsilon(U_{b_2 - \rho_2}^{*,2}(\vec{f}))(x_0) \\ &\leq C \|b_1\|_{BMO} M_\varepsilon(U_{b_2}^{*,2}(\vec{f}))(x_0). \end{aligned}$$

Similarly,

$$T_3 \leq C \|b_2\|_{BMO} M_\varepsilon(U_{b_1 - \rho_1}^*(\vec{f}))(x_0) \leq C \|b_2\|_{BMO} M_\varepsilon(U_{b_1}^*(\vec{f}))(x_0).$$

For T_4 , we denote that $f_i^0 = f_i \chi_{3Q}$ and $f_i^\infty = f_i - f_i^0$. Note that $c = \sup_\eta |\sum_{j=1}^3 c_j|$, where

$$c_1 = U_\eta((b_1 - \rho_1)f_1^0, (b_2 - \rho_2)f_2^\infty)(x_0),$$

$$c_2 = U_\eta((b_1 - \rho_1)f_1^\infty, (b_2 - \rho_2)f_2^0)(x_0),$$

$$c_3 = U_\eta((b_1 - \rho_1)f_1^\infty, (b_2 - \rho_2)f_2^\infty)(x_0).$$

we may split it in the following way

$$T_4 \leq T_{4,1} + T_{4,2} + T_{4,3} + T_{4,4},$$

where

$$\begin{aligned}
T_{4,1} &= \left(\frac{1}{|Q|} \int_Q \sup_{\eta>0} |U_\eta((b_1 - \rho_1)f_1^0, (b_2 - \rho_2)f_2^0)(z)|^\delta dz \right)^{\frac{1}{\delta}}, \\
T_{4,2} &= \left(\frac{1}{|Q|} \int_Q \sup_{\eta>0} |U_\eta((b_1 - \rho_1)f_1^0, (b_2 - \rho_2)f_2^\infty)(z) - U_\eta((b_1 - \rho_1)f_1^0, (b_2 - \rho_2)f_2^\infty)(x_0)|^\delta dz \right)^{\frac{1}{\delta}}, \\
T_{4,3} &= \left(\frac{1}{|Q|} \int_Q \sup_{\eta>0} |U_\eta((b_1 - \rho_1)f_1^\infty, (b_2 - \rho_2)f_2^0)(z) - U_\eta((b_1 - \rho_1)f_1^\infty, (b_2 - \rho_2)f_2^0)(x_0)|^\delta dz \right)^{\frac{1}{\delta}} \\
&\text{and} \\
T_{4,4} &= \left(\frac{1}{|Q|} \int_Q \sup_{\eta>0} |U_\eta((b_1 - \rho_1)f_1^\infty, (b_2 - \rho_2)f_2^\infty)(z) - U_\eta((b_1 - \rho_1)f_1^\infty, (b_2 - \rho_2)f_2^\infty)(x_0)|^\delta dz \right)^{\frac{1}{\delta}}.
\end{aligned}$$

We consider the first term. Use the Kolmogorov's inequality, lemma 2.2 (a), Theorem B with $w_i \equiv 1$ for $m = 2$ and (2.6), then we deduce that

$$\begin{aligned}
T_{4,1} &\leq C \left(\frac{1}{|Q|} \int_Q |U^*((b_1 - \rho_1)f_1^0, (b_2 - \rho_2)f_2^0)(z)|^{p_0\delta} dz \right)^{1/p_0\delta} \\
&\leq C |Q|^{-2} \|U^*((b_1 - \rho_1)f_1^0, (b_2 - \rho_2)f_2^0)\|_{L^{\frac{1}{2}, \infty}(Q)} \\
&\leq C |Q|^{-2} \|(b_1 - \rho_1)\|_{L^1(Q)} \|f_1^0\|_{L^1(Q)} \|(b_2 - \rho_2)f_2^0\|_{L^1(Q)} \\
&\leq C \|b_1\|_{BMO} \|f_1^0\|_{L(\log L)} \|b_2\|_{BMO} \|f_2^0\|_{L(\log L)} \\
&\leq C \prod_{i=1}^m \|b_i\|_{BMO} \mathcal{M}_{L(\log L)}(\vec{f})(x_0).
\end{aligned}$$

By mean value theorem we deduce

$$\begin{aligned}
T_{4,2} &\leq \frac{C}{|Q|} \int_Q \sup_{\eta>0} \left| U_\eta((b_1 - \rho_1)f_1^0, (b_2 - \rho_2)f_2^\infty)(z) - U_\eta((b_1 - \rho_1)f_1^0, (b_2 - \rho_2)f_2^\infty)(x_0) \right| dz \\
&\leq C \frac{1}{|Q|} \int_Q \int_{3Q} |(b_1 - \rho_1)f_1(y_1)| dy_1 \int_{(3Q)^c} \frac{|x_0 - z|^\varepsilon |b_2(y_2) - \rho_2| |f_2(y_2)| dy_2}{(|z - y_1| + |z - y_2|)^{2n+\varepsilon}} dz \\
&\leq C \sum_{j=1}^{\infty} \frac{j|Q|^{\varepsilon/n}}{(3^j|Q|^{1/n})^{2n+\varepsilon}} \int_{3^{j+1}Q} |(b_1 - \rho_1)f_1(y_1)| dy_1 \int_{3^{j+1}Q} |b_2(y_2) - \rho_2| |f_2(y_2)| dy_2 \\
&\leq C \sum_{j=1}^{\infty} \frac{1}{3^{j\varepsilon}} \prod_{i=1}^2 \|b_i\|_{BMO} \|f_i\|_{L(\log L), 3^{j+1}Q} \\
&\leq C \prod_{i=1}^2 \|b_i\|_{BMO} \mathcal{M}_{L(\log L)}(\vec{f})(x_0).
\end{aligned}$$

Similarly as $T_{4,2}$, we can get the estimates for $T_{4,3}$. Now we are in a position to deal $T_{4,4}$. Note that

$$\left| U_\eta((b_1 - \rho_1)f_1^\infty, (b_2 - \rho_2)f_2^\infty)(z) - (|U_\eta((b_1 - \rho_1)f_1^\infty, (b_2 - \rho_2)f_2^\infty))(x_0) \right|$$

$$\begin{aligned}
&\leq C \int_{(\mathbb{R}^n \setminus 3Q)^2} \frac{|Q|^{\frac{\varepsilon}{n}} |(b_1 - \rho_1)| |b_2(y_2) - \rho_2|}{|(x_0 - y_1, x_0 - y_2)|^{2n+\varepsilon}} \prod_{i=1}^2 |f_i^\infty(z_i)| d\vec{y} \\
&\leq C \sum_{k=1}^{\infty} \int_{(3^{k+1}Q)^2 \setminus (3^kQ)^2} \frac{|Q|^{\frac{\varepsilon}{n}} |(b_1 - \rho_1)| |b_2(y_2) - \rho_2|}{(3^k |Q|^{\frac{1}{n}})^{2n+\varepsilon}} \prod_{i=1}^2 |f_i^\infty(y_i)| d\vec{y} \\
&\leq C \prod_{i=1}^2 \|b_i\|_{BMO} \mathcal{M}_{L(\log L)}(\vec{f})(x_0).
\end{aligned}$$

Thus, we have

$$T_{4,4} \leq C \prod_{i=1}^2 \|b_i\|_{BMO} \mathcal{M}_{L(\log L)}(\vec{f})(x_0).$$

Thus we complete the proof of this lemma for $U_{\Pi b}^*(\vec{f})$.

Next, we prove (3.3) for $I_{\alpha, \Pi b}(\vec{f})$, we split

$$\begin{aligned}
I_{\alpha, \Pi b}(\vec{f})(x) &= (b_1(x) - \rho_1)(b_2(x) - \rho_2) I_{\alpha}(\vec{f})(x) - (b_1(x) - \rho_1) I_{\alpha}(f_1, (b_2 - \rho_2)f_2)(x) \\
&\quad - (b_2(x) - \rho_2) I_{\alpha}((b_1 - \rho_1)f_1, f_2)(x) + I_{\alpha}((b_1 - \rho_1)f_1, (b_2 - \rho_2)f_2)(x) \\
&= -(b_1(x) - \rho_1)(b_2(x) - \rho_2) I_{\alpha}(\vec{f})(x) + (b_1(x) - \rho_1) I_{b_2 - \rho_2, \alpha}^2(f_1, f_2)(x) \\
&\quad + (b_2(x) - \rho_2) I_{b_1 - \rho_1, \alpha}^1(f_1, f_2)(x) + I_{\alpha}((b_1 - \rho_1)f_1, (b_2 - \rho_2)f_2)(x).
\end{aligned}$$

Fix $x_0 \in \mathbb{R}^n$ and let Q be a cube centered at x_0 . Denote any constants by $c = (I_{\alpha}(f_1^0, (b_2 - \rho_2)f_2^\infty)(x_0) + I_{\alpha}(f_1^\infty, (b_2 - \rho_2)f_2^0)(x_0) + I_{\alpha}(f_1^\infty, (b_2 - \rho_2)f_2^\infty)(x_0)) =: c_1 + c_2 + c_3$. Then

Since $0 < \delta \leq \frac{1}{m}$, then we have

$$\left(\frac{1}{|Q|} \int_Q |I_{\alpha, \Pi b}(\vec{f})(z)|^\delta - |c|^\delta dz \right)^{\frac{1}{\delta}} \leq C(S_1 + S_2 + S_3 + S_4),$$

where

$$S_1 = \left(\frac{1}{|Q|} \int_Q |(b_1(z) - \rho_1)(b_2(z) - \rho_2)|^\delta |I_{\alpha}(\vec{f})(z)|^\delta dz \right)^{\frac{1}{\delta}},$$

$$S_2 = \left(\frac{1}{|Q|} \int_Q |(b_1(x) - \rho_1) I_{b_2 - \rho_2, \alpha}^2(f_1, f_2)(z)|^\delta dz \right)^{\frac{1}{\delta}}.$$

$$S_3 = \left(\frac{1}{|Q|} \int_Q |(b_2(x) - \rho_2) I_{b_1 - \rho_1, \alpha}^1(f_1, f_2)(z)|^\delta dz \right)^{\frac{1}{\delta}}$$

and

$$S_4 = \left(\frac{1}{|Q|} \int_Q |I_{\alpha}((b_1 - \rho_1)f_1, (b_2 - \rho_2)f_2)(z) - c|^\delta dz \right)^{\frac{1}{\delta}}.$$

Let $\rho_j = (b_j)_{3Q}$ be the average of b_j on $3Q$ for $j = 1, 2$.

For any $1 < r_1, r_2, r_3 < \infty$ with $\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} = 1$ and $r_3 < \frac{\varepsilon}{\delta}$, S_1 can be estimated by using the Holder's inequality and (2.7).

$$\begin{aligned} S_1 &\leq C \left(\frac{1}{|Q|} \int_Q |b_1(z) - \rho_1|^{\delta r_1} dz \right)^{\frac{1}{\delta r_1}} \left(\frac{1}{|Q|} \int_Q |b_2(z) - \rho_2|^{\delta r_2} dz \right)^{\frac{1}{\delta r_2}} \\ &\quad \times \left(\frac{1}{|Q|} \int_Q |I_\alpha(\vec{f})(z)|^{\delta r_3} dz \right)^{\frac{1}{\delta r_3}} \\ &\leq C \prod_{j=1}^2 \|b_j\|_{BMO} M_\varepsilon(I_\alpha(\vec{f}))(x_0). \end{aligned}$$

As the argument of T_2 , we still take $1 < t_1, t_2 < \infty$ with $1 = 1/t_1 + 1/t_2$ and $t_2 < \frac{\varepsilon}{\delta}$

$$\begin{aligned} S_2 &= \left(\frac{1}{|Q|} \int_Q |(b_1(x) - \rho_1) I_{b_2-\rho_2, \alpha}^2(f_1, f_2)(z)|^\delta dz \right)^{\frac{1}{\delta}} \\ &\leq C \|b_1\|_{BMO} \mathcal{M}_{t_2 \delta}(I_{b_2-\rho_2, \alpha}^2(f_1, f_2))(x_0) \\ &\leq C \|b_1\|_{BMO} \mathcal{M}_\varepsilon(I_{b_2-\rho_2, \alpha}^2(f_1, f_2))(x_0). \end{aligned}$$

Similarly, we can get the estimates for S_3 as we deal S_2 . Next, for S_4 , we denote that $f_i^0 = f_i \chi_{3Q}$ and $f_i^\infty = f_i - f_i^0$ and Let $c = (I_\alpha(f_1^0, (b_2 - \rho_2)f_2^\infty)(x_0) + I_\alpha(f_1^\infty, (b_2 - \rho_2)f_2^0)(x_0) + I_\alpha(f_1^\infty, (b_2 - \rho_2)f_2^\infty)(x_0))$, then S_4 can be written as

$$S_4 \leq S_{4,1} + S_{4,2} + S_{4,3} + S_{4,4},$$

where

$$S_{4,1} = \left(\frac{1}{|Q|} \int_Q |I_\alpha((b_1 - \rho_1)f_1^0, (b_2 - \rho_2)f_2^0)(z)|^\delta dz \right)^{\frac{1}{\delta}},$$

$$S_{4,2} = \left(\frac{1}{|Q|} \int_Q |I_\alpha((b_1 - \rho_1)f_1^0, (b_2 - \rho_2)f_2^\infty)(z) - I_\alpha((b_1 - \rho_1)f_1^0, (b_2 - \rho_2)f_2^\infty)(x_0)|^\delta dz \right)^{\frac{1}{\delta}},$$

$$S_{4,3} = \left(\frac{1}{|Q|} \int_Q |I_\alpha((b_1 - \rho_1)f_1^\infty, (b_2 - \rho_2)f_2^0)(z) - I_\alpha((b_1 - \rho_1)f_1^\infty, (b_2 - \rho_2)f_2^0)(x_0)|^\delta dz \right)^{\frac{1}{\delta}}$$

and

$$S_{4,4} = \left(\frac{1}{|Q|} \int_Q |I_\alpha((b_1 - \rho_1)f_1^\infty, (b_2 - \rho_2)f_2^\infty)(z) - I_\alpha((b_1 - \rho_1)f_1^\infty, (b_2 - \rho_2)f_2^\infty)(x_0)|^\delta dz \right)^{\frac{1}{\delta}}.$$

Use Hölder inequality, the Kolmogorov's inequality (2.9) when $p = \frac{1}{2}$ and $q = \frac{n}{2n-\alpha}$, (2.11) in

Lemma 2.4, then we deduce that

$$\begin{aligned}
S_{4,1} &\leq C \left(\frac{1}{|Q|} \int_Q \left| I_\alpha((b_1 - \rho_1)f_1^0, (b_2 - \rho_2)f_2^0)(z) \right|^{\frac{1}{2}} dz \right)^2 \\
&\leq C |Q|^{\frac{\alpha}{n}-2} \|I_\alpha((b_1 - \rho_1)f_1^0, (b_2 - \rho_2)f_2^0)\|_{L^{\frac{n}{2n-\alpha}, \infty}(Q)} \\
&\leq C |Q|^{\frac{\alpha}{n}-2} \|(b_1 - \rho_1)f_1^0\|_{L^1(Q)} \|(b_2 - \rho_2)f_2^0\|_{L^1(Q)} \\
&\leq C |3Q|^{\frac{\alpha}{n}} \|b_1\|_{BMO} \|f_1^0\|_{L(\log L), Q} \|b_2\|_{BMO} \|f_2^0\|_{L(\log L), Q} \\
&\leq C \prod_{j=1}^2 \|b_j\|_{BMO} \mathcal{M}_{L(\log L), \alpha}(\vec{f})(x_0).
\end{aligned}$$

By mean value theorem again, we deduce

$$\begin{aligned}
S_{4,2} &\leq \frac{C}{|Q|} \int_Q \left| I_\alpha((b_1 - \rho_1)f_1^0, (b_2 - \rho_2)f_2^\infty)(z) - I_\alpha((b_1 - \rho_1)f_1^0, (b_2 - \rho_2)f_2^\infty)(x_0) \right| dz \\
&\leq C \frac{1}{|Q|} \int_{3Q} |(b_1(y_1) - \rho_1)f_1(y_1)| dy_1 \int_{(3Q)^c} \frac{|x_0 - z| |(b_1 - \rho_1)| |b_2(y_2) - \rho_2| |f_2(y_2)| dy_2}{(|z_1 - y_1| + |z_2 - y_2|)^{2n-\alpha+1}} dz \\
&\leq C \sum_{j=1}^{\infty} \frac{j}{(3^j |Q|^{1/n})^{2n-\alpha+1}} \int_{3Q} |(b_1(y_1) - \rho_1)f_1(y_1)| dy_1 \int_{3^{j+1}Q} |b_2(y_2) - \rho_2| |f_2(y_2)| dy_2 \\
&\leq C \prod_{j=1}^2 \|b_j\|_{BMO} \mathcal{M}_{L(\log L), \alpha}(\vec{f})(x_0).
\end{aligned}$$

Similarly as $S_{4,2}$, we can get the estimates for $S_{4,3}$. Now we are in a position to deal $S_{4,4}$.

$$\begin{aligned}
&\left| I_\alpha((b_1 - \rho_1)f_1^\infty, (b_2 - \rho_2)f_2^\infty)(z) - (I_\alpha((b_1 - \rho_1)f_1^\infty, (b_2 - \rho_2)f_2^\infty))(x_0) \right| \\
&\leq C \int_{(\mathbb{R}^n \setminus 3Q)^2} \frac{|Q|^{\frac{1}{n}} |b_1(y_1) - \rho_1| |b_2(y_2) - \rho_2|}{|(x_0 - y_1, x_0 - y_2)|^{2n-\alpha+1}} \prod_{i=1}^2 |f_i^\infty(y_i)| d\vec{y} \\
&\leq C \sum_{k=1}^{\infty} \int_{(3^{k+1}Q)^2 \setminus (3^kQ)^2} \frac{|Q|^{\frac{1}{n}} |b_1(y_1) - \rho_1| |b_2(y_2) - \rho_2|}{(3^k |Q|^{\frac{1}{n}})^{2n-\alpha+1}} \prod_{i=1}^2 |f_i^\infty(y_i)| d\vec{y} \\
&\leq C \sum_{k=1}^{\infty} \frac{k}{3^k} \|b_2\|_{BMO} |3^{k+1}Q|^{\frac{\alpha}{n}} \prod_{j=1}^2 \|f_j^\infty\|_{L(\log L), 3^{k+1}Q} \\
&\leq C \prod_{j=1}^2 \|b_j\|_{BMO} \mathcal{M}_{L(\log L), \alpha}(\vec{f})(x_0).
\end{aligned}$$

So we obtain

$$S_{4,i} \leq C \|b_1\|_{BMO} \|b_2\|_{BMO} \mathcal{M}_{L(\log L), \alpha}(\vec{f})(x_0).$$

Thus we complete the proof for this lemma.

Proposition 3.2 (Pointwise control of $M_\delta^\sharp(U^*(\vec{f})), M_\delta^\sharp(V^*(\vec{f})), M_\delta^\sharp(I_\alpha(\vec{f}))$) Let $0 < \delta < \varepsilon$, $0 < \delta < \frac{1}{m}$ and $0 < \alpha < mn$. Then there is $C > 0$ depending on δ and ε such that

$$M_\delta^\sharp(U^*(\vec{f}))(x) \leq C \mathcal{M}(f)(x), \quad (3.4)$$

$$M_\delta^\sharp(V^*(\vec{f}))(x) \leq C\mathcal{M}(f)(x), \quad (3.5)$$

$$M_{1/m}^\sharp(I_\alpha(\vec{f}))(x) \leq \mathcal{M}_\alpha(f)(x). \quad (3.6)$$

for all bounded \vec{f} with compact support.

Proof.

The proof of (3.4) and (3.5) follows from similar steps in Theorem 3.2 of [19] and combine the method we used in the above proposition, here we omit the proof. On the other hand, (2.7) has already been obtained in [4], Proposition 5.2.

Now, we can obtain

Theorem 3.1 *Let $0 < p$ and $w \in A_\infty$. Suppose that $\vec{b} \in (BMO)^m$. Then there is a constant C independent of \vec{b} and a constant C_1 (may dependent on \vec{b}) such that*

$$\int_{\mathbb{R}^n} |U_{\Pi b}^*(\vec{f})(x)|^p \omega(x) dx \leq C \prod_{i=1}^m \|b_i\|_{BMO} \int_{\mathbb{R}^n} [\mathcal{M}_{L(\log L)}(f)(x)]^p w(x) dx, \quad (3.7)$$

$$\begin{aligned} \sup_{t>0} \frac{1}{\Phi^m(1/t)} w(\{y \in \mathbb{R}^n : |U_{\Pi b}^* \vec{f}(y)| > t^m\}) \\ \leq C_1 \sup_{t>0} \frac{1}{\Phi^m(1/t)} w(\{y \in \mathbb{R}^n : \mathcal{M}_{L(\log L)}(f)(y) > t^m\}). \end{aligned} \quad (3.8)$$

Similar results hold for $V_{\Pi b}^*(\vec{f})$.

Proof of Theorem 3.1. The proof of the above Theorem 3.1 are now standard as the case for multilinear C-Z singular integral operators. We briefly indicate such arguments in the case $m=2$, but, as the reader will immediately notice, and iterative procedure using (3.1) and (3.2) can be followed to obtain the general case.

Using Fefferman-Stein inequality and pointwise estimate in proposition 3.1 we will have

$$\begin{aligned} \|U_{\Pi b}^*(\vec{f})\|_{L^p(\omega)} &\leq \|M_\delta(U_{\Pi b}^*(\vec{f}))\|_{L^p(\omega)} \leq C \|M_\delta^\sharp(U_{\Pi b}^*(\vec{f}))\|_{L^p(\omega)} \\ &\leq C \prod_{i=1}^2 \|b_i\|_{BMO} \left(\|\mathcal{M}_{L(\log L)}(\vec{f})\|_{L^p(\omega)} + \|M_\varepsilon^\sharp(U^*(\vec{f}))\|_{L^p(\omega)} \right) \\ &\quad + C \left(\|b_2\|_{BMO} \|M_\varepsilon^\sharp(U_{b_1}^*(\vec{f}))\|_{L^p(\omega)} + \|b_1\|_{BMO} \|M_\varepsilon^\sharp(U_{b_2}^*(\vec{f}))\|_{L^p(\omega)} \right). \end{aligned}$$

Hence, next we estimate $\|M_\varepsilon^\sharp(U_{b_2}^*(\vec{f}))\|_{L^p(\omega)}$, $\|M_\varepsilon^\sharp(U_{b_1}^*(\vec{f}))\|_{L^p(\omega)}$ has the similar estimate. Set $c_\eta = U_\eta(f_1^0, (b_2 - \rho_2)f_2^\infty)(x_0) + U_\eta(f_1^\infty, (b_2 - \rho_2)f_2^0)(x_0) + U_\eta(f_1^\infty, (b_2 - \rho_2)f_2^\infty)(x_0)$ and $c = \sup_{\eta>0} \{|c_\eta|\}$, then

$$\begin{aligned} |U_{b_2}^*(\vec{f})(z) - c| &\leq \sup_{\eta>0} \left| \int_{(\mathbb{R}^n)^2} K_{u,\eta}(z, y_1, y_2) ((b_2(z) - \rho_2) - (b_2(y_2) - \rho_2)) \prod_{i=1}^2 f_i(y_i) d\vec{y} + c_\eta \right| \\ &\leq C |b_2(z) - \rho_2| |U^*(f_1, f_2)(z)| + \sup_{\eta>0} |U_\eta(f_1, (b_2 - \rho_2)f_2)(z) - c_\eta|. \end{aligned}$$

For arbitrary $0 < \varepsilon' < \frac{1}{2}$, take $1 < t_1, t_2 < \infty$ with $1 = 1/t_1 + 1/t_2$ and $t_2 < \frac{\varepsilon'}{\varepsilon}$, we have

$$\begin{aligned} & \left(\frac{1}{|Q|} \int_Q |(b_2(z) - \rho_2)U^*(f_1, f_2)(z)|^\varepsilon dz \right)^{\frac{1}{\varepsilon}} \\ & \leq \left(\frac{1}{|Q|} \int_Q |b_2(z) - \rho_2|^{t_1\varepsilon} dz \right)^{\frac{1}{t_1\varepsilon}} \left(\frac{1}{|Q|} \int_Q |U^*(f_1, f_2)(z)|^{t_2\varepsilon} dz \right)^{\frac{1}{t_2\varepsilon}} \\ & \leq C \|b_2\|_{BMO} \mathcal{M}_{\varepsilon'}(U^*(f_1, f_2))(x_0). \end{aligned}$$

As the proof of Proposition 3.1, then $U_\eta(f_1, (b_2 - \rho_2)f_2)$ can be written as

$$\begin{aligned} U_\eta(f_1, (b_2 - \rho_2)f_2) &= U_\eta(f_1^0, (b_2 - \rho_2)f_2^0) + U_\eta(f_1^0, (b_2 - \rho_2)f_2^\infty) \\ &\quad + U_\eta(f_1^\infty, (b_2 - \rho_2)f_2^0) + U_\eta(f_1^\infty, (b_2 - \rho_2)f_2^\infty). \end{aligned}$$

Take $1 < p_0 < 1/(2\varepsilon)$ and using Hölder's inequality again, we have

$$\begin{aligned} & \left(\frac{1}{|Q|} \int_Q \sup_{\eta>0} |U_\eta(f_1, (b_2 - \rho_2)f_2)(z) - c_\eta|^\varepsilon dz \right)^{\frac{1}{\varepsilon}} \\ & \leq \left(\frac{1}{|Q|} \int_Q \sup_{\eta>0} \left| U_\eta(f_1, (b_2 - \rho_2)f_2)(z) - c_\eta \right|^{p_0\varepsilon} dz \right)^{1/p_0\varepsilon} \\ & \leq (G_1 + G_2 + G_3 + G_4), \end{aligned}$$

where

$$\begin{aligned} G_1 &= \left(\frac{1}{|Q|} \int_Q \sup_{\eta>0} \left| U_\eta(f_1^0, (b_2 - \rho_2)f_2^0)(z) \right|^{p_0\varepsilon} dz \right)^{1/p_0\varepsilon}, \\ G_2 &= \left(\frac{1}{|Q|} \int_Q \sup_{\eta>0} \left| U_\eta(f_1^0, (b_2 - \rho_2)f_2^\infty)(z) - U_\eta(f_1^0, (b_2 - \rho_2)f_2^\infty)(x_0) \right|^{p_0\varepsilon} dz \right)^{1/p_0\varepsilon}, \\ G_3 &= \left(\frac{1}{|Q|} \int_Q \sup_{\eta>0} \left| U_\eta(f_1^\infty, (b_2 - \rho_2)f_2^0)(z) - U_\eta(f_1^\infty, (b_2 - \rho_2)f_2^0)(x_0) \right|^{p_0\varepsilon} dz \right)^{1/p_0\varepsilon} \end{aligned}$$

and

$$G_4 = \left(\frac{1}{|Q|} \int_Q \sup_{\eta>0} \left| U_\eta(f_1^\infty, (b_2 - \rho_2)f_2^\infty)(z) - U_\eta(f_1^\infty, (b_2 - \rho_2)f_2^\infty)(x_0) \right|^{p_0\varepsilon} dz \right)^{1/p_0\varepsilon}.$$

The similar procedure for T_4 in the Proposition 3.1, we obtain

$$G_1 \leq C \|b_2\|_{BMO} \mathcal{M}_{L(\log L)}^2(\vec{f})(x_0).$$

By mean value theorem we deduce

$$G_2 \leq C \|b_2\|_{BMO} \mathcal{M}_{L(\log L)}^2(\vec{f})(x_0).$$

Similarly as G_2 , we can get the estimates for G_3 . Moreover

$$G_4 \leq C \|b_2\|_{BMO} \mathcal{M}_{L(\log L)}^2(\vec{f})(x_0).$$

By proposition 3.2, so we have

$$\begin{aligned} \|M_{\varepsilon}^{\sharp}[U_{b_2}^*(\vec{f})]\|_{L^p(\omega)} &\leq C\|b_2\|_{BMO}(\|\mathcal{M}(\vec{f})\|_{L^p(\omega)} + \|\mathcal{M}_{L(\log L)}^2(\vec{f})\|_{L^p(\omega)}) \\ &\leq C\|b_2\|_{BMO}\|\mathcal{M}_{L(\log L)}(\vec{f})\|_{L^p(\omega)}. \end{aligned}$$

The desired inequality now follows. Since the left main steps and the ideas are almost the same as [25], here we omit the proof. So we get the estimate of strong type and weak type.

Proof of Theorem 1.1-1.2. Theorem 1.1 follows by the reason that $T_{*,\Pi b}(\vec{f}) \leq U_{\Pi b}^*(\vec{f})(x) + V_{\Pi b}^*(\vec{f})(x)$, Theorem 3.1 and the weighted strong boundedness of $\mathcal{M}_{L(\log L)}$ in [19]. Theorem 1.2 follows by repeating the same steps as in [19], [25] and the method used in [29]. Since the main steps and the ideas are almost the same, here we omit the proof.

Proof of Theorem 1.3. Theorem 1.3 follows by using Proposition 3.1 and the estimate for $I_{b,\alpha}^j$ ($j = 1, 2$), which is Theorem 2.7 in [4].

4 Weighted end-point estimates for $I_{\alpha,\Pi b}(\vec{f})$

Firstly, we will consider the end-point estimate of multilinear fractional $L(\log L)$ type maximal operator.

Proposition 4.1 (Weighted end-point estimate for $\mathcal{M}_{L(\log L),\alpha}$) *Let $\Phi(t) = t(1 + \log^+ t)$ and $\vec{\omega} \in A_{((1,\dots,1), \frac{n}{mn-\alpha})}$. If $0 < \alpha < mn$, then there is a $C > 0$ such that*

$$\begin{aligned} &\nu_{\vec{\omega}}^{\frac{n}{mn-\alpha}} \left(\left\{ x \in \mathbb{R}^n : \mathcal{M}_{L(\log L),\alpha}(\vec{f})(x) > t^{\frac{mn-\alpha}{n}} \right\} \right) \\ &\leq C \left\{ \left[1 + \frac{\alpha}{mn} \log^+ \left(\prod_{i=1}^m \int_{\mathbb{R}^n} \Phi^{(m)} \left(\frac{|f_i(y_i)|}{t} \right) dy_i \right) \right]^m \prod_{j=1}^m \int_{\mathbb{R}^n} \Phi^{(m)} \left(\frac{|f_j(y_j)|}{t} \right) \omega_j(y_j) dy_j \right\}^{\frac{n}{mn-\alpha}}. \end{aligned} \quad (4.1)$$

If $0 < \alpha_j < n$ for each $1 \leq j \leq m$, $\sum_{j=1}^m \alpha_j = \alpha$, then there is a $C > 0$ such that

$$\begin{aligned} &\nu_{\vec{\omega}}^{\frac{n}{mn-\alpha}} \left(\left\{ x \in \mathbb{R}^n : \mathcal{M}_{L(\log L),\alpha}(\vec{f})(x) > t^{\frac{mn-\alpha}{n}} \right\} \right) \\ &\leq C \left\{ \prod_{j=1}^m \left[1 + \frac{\alpha_j}{n} \log^+ \left(\prod_{i=1}^m \int_{\mathbb{R}^n} \Phi^{(m)} \left(\frac{|f_i(y_i)|}{t} \right) dy_i \right) \right] \int_{\mathbb{R}^n} \Phi^{(m)} \left(\frac{|f_j(y_j)|}{t} \right) \omega_j(y_j) dy_j \right\}^{\frac{n}{mn-\alpha}}. \end{aligned} \quad (4.2)$$

Proof. By the homogeneity, we can assume $t = 1$. We first prove (4.2). Denote that

$$E_1 = \left\{ x \in \mathbb{R}^n : \mathcal{M}_{L(\log L),\alpha}(\vec{f})(x) > 1 \right\} \text{ and } E_{1,k} = E_1 \cap B(0, k),$$

where $B(0, k) = \{x \in \mathbb{R}^n : |x| \leq k\}$. By the monotone convergence theorem, it suffices to estimate $E_{1,k}$.

For any $x \in E_{1,k}$, there is a cube Q_x such that

$$1 < |Q_x|^{\frac{\alpha}{n}} \prod_{j=1}^m \|f_j\|_{L(\log L), Q} . \quad (4.3)$$

Hence, $\{Q_x\}_{x \in E_{1,k}}$ is a family of cubes covering $E_{1,k}$. Using a covering argument, we obtain a finite family of disjoint cubes $\{Q_{x_l}\}$ whose dilations cover F such that

$$|E_{1,k}| \leq C \sum_l |Q_{x_l}| \quad \text{and} \quad 1 < |Q_{x_l}|^{\frac{\alpha}{n}} \prod_{j=1}^m \|f_j\|_{L(\log L), Q_{x_l}} . \quad (4.4)$$

We follow the main steps first as in [25] and denote C_h^m to be the family of all subset $\sigma = (\sigma(1), \dots, \sigma(h))$ from $\{1, \dots, m\}$ with $1 \leq h \leq m$ different elements. Given $\sigma \in C_h^m$ and a cube Q_{x_l} , if $|Q_{x_l}|^{\alpha_{\sigma(j)}} \|f_{\sigma(j)}\|_{L(\log L), Q_{x_l}} > 1$ for $j = 1, \dots, h$, we say that $j \in B_\sigma$ and $|Q_{x_l}|^{\alpha_{\sigma(j)}} \|f_{\sigma(j)}\|_{L(\log L), Q_{x_l}} \leq 1$ for $j = h+1, \dots, m$. Denote

$$A_k = \prod_{j=1}^k |Q_{x_l}|^{\alpha_{\sigma(j)}/n} \|f_{\sigma(j)}\|_{L(\log L), Q_{x_l}}$$

and $A_0 = 1$. Then it is easy to check that if $\sigma \in C_h^m$ and $j \in B_\sigma$, for any $1 \leq k \leq m$, we have $A_k > 1$ and

$$\begin{aligned} 1 &< \prod_{j=1}^k |Q_{x_l}|^{\alpha_{\sigma(j)}/n} \|f_{\sigma(j)}\|_{L(\log L), Q_{x_l}} \\ &= \left\| |Q_{x_l}|^{\alpha_{\sigma(k)}/n} f_{\sigma(k)} A_{k-1} \right\|_{\Phi, Q_{x_l}} . \end{aligned}$$

Or, equivalently

$$1 < \frac{1}{|Q_{x_l}|} \int_{Q_{x_l}} \Phi \left(|Q_{x_l}|^{\alpha_{\sigma(k)}/n} f_{\sigma(k)} \left(\prod_{j=1}^{k-1} |Q_{x_l}|^{\alpha_{\sigma(j)}/n} \|f_{\sigma(j)}\|_{L(\log L), Q_{x_l}} \right) \right) . \quad (4.5)$$

By the following equivalence

$$\|f\|_{\Phi, Q} \simeq \inf_{\mu > 0} \left\{ \mu + \frac{\mu}{|Q_{x_l}|} \int_{Q_{x_l}} \Phi(|f|/\mu) \right\} .$$

If $1 \leq j \leq m-h-1$, we obtain

$$\Phi^j(A_{m-j}) = \Phi^j(\| |Q_{x_l}|^{\alpha_{\sigma(m-j)}/n} f_{\sigma(m-j)} A_{m-j-1} \|_{\Phi, Q_{x_l}}) .$$

Since $\| |Q_{x_l}|^{\alpha_{\sigma(m-j)}/n} f_{\sigma(m-j)} A_{m-j-1} \|_{\Phi, Q} > 1$, Using the fact that Φ is submultiplicative (i.e. $\Phi(st) \leq \Phi(s)\Phi(t)$ for $s, t > 0$) and Jensen's inequality, we have

$$\begin{aligned} \Phi^j(A_{m-j}) &= \Phi^j(\| |Q_{x_l}|^{\alpha_{\sigma(m-j)}/n} f_{\sigma(m-j)} A_{m-j-1} \|_{\Phi, Q}) \\ &\leq C \Phi^j \left(1 + \frac{1}{|Q_{x_l}|} \int_Q \Phi(|Q_{x_l}|^{\alpha_{\sigma(m-j)}/n} f_{\sigma(m-j)} A_{m-j-1}) \right) \\ &\leq C \frac{1}{|Q_{x_l}|} \int_Q \Phi^{j+1}(|Q_{x_l}|^{\alpha_{\sigma(m-j)}/n} f_{\sigma(m-j)}) \Phi^{j+1}(A_{m-j-1}) . \end{aligned}$$

By iterating the inequalities above and the fact that $\| |Q_{x_l}|^{\alpha_{\sigma(j)}/n} f_{\sigma(j)} \|_{\Phi, Q_{x_l}} > 1$ for $j \in B_\sigma$, $\Phi^{j+1} \leq \Phi^m$ and $\Phi^{m-h+1} \leq \Phi^m$ for $1 \leq h \leq m$ and $0 \leq j \leq m-h-1$, we have

$$\begin{aligned}
1 &< \frac{1}{|Q_{x_l}|} \int_{Q_{x_l}} \Phi(|Q_{x_l}|^{\alpha_{\sigma(m)}/n} f_{\sigma(m)}) \frac{1}{|Q_{x_l}|} \int_{Q_{x_l}} \Phi^2(|Q_{x_l}|^{\alpha_{\sigma(m-1)}/n} f_{\sigma(m-1)}) \Phi^2(A_{m-2}) \\
&\leq \left(\prod_{j=0}^{m-h-1} \frac{1}{|Q_{x_l}|} \int_{Q_{x_l}} \Phi^{j+1}(|Q_{x_l}|^{\alpha_{\sigma(m-1)}/n} f_{\sigma(m-1)}) \right) \prod_{j=1}^h \Phi^{m-h}(\| |Q_{x_l}|^{\alpha_{\sigma(j)}/n} f_{\sigma(j)} \|_{\Phi, Q_{x_l}}) \\
&\leq \left(\prod_{j=0}^{m-h-1} \frac{1}{|Q_{x_l}|} \int_{Q_{x_l}} \Phi^{j+1}(|Q_{x_l}|^{\alpha_{\sigma(m-1)}/n} f_{\sigma(m-1)}) \right) \prod_{j=1}^h \frac{1}{|Q_{x_l}|} \int_{Q_{x_l}} \Phi^{m-h+1}(|Q_{x_l}|^{\alpha_{\sigma(j)}/n} f_{\sigma(j)}) \\
&\leq C \prod_{j=1}^m \frac{1}{|Q_{x_l}|} \int_{Q_{x_l}} \Phi^m(|Q_{x_l}|^{\frac{\alpha_j}{n}} f_j).
\end{aligned} \tag{4.6}$$

We obtain

$$\begin{aligned}
1 &< C \prod_{j=1}^m \frac{1}{|Q_{x_l}|} \int_{Q_{x_l}} \Phi^m(|Q_{x_l}|^{\frac{\alpha_j}{n}} f_j) \\
&\leq C \prod_{j=1}^m \frac{1}{|Q_{x_l}|} \int_{Q_{x_l}} \Phi^m(|Q_{x_l}|^{\frac{\alpha_j}{n}}) \Phi^m(f_j) \\
&\leq C \prod_{j=1}^m \frac{1}{|Q_{x_l}|} |Q_{x_l}|^{\frac{\alpha_j}{n}} (1 + \log^+ |Q_{x_l}|^{\frac{\alpha_j}{n}}) \int_{Q_{x_l}} \Phi^m(f_j).
\end{aligned} \tag{4.7}$$

Since $\alpha_j < n$, there exists a constant $C_0 > 1$ and η_1, \dots, η_m small enough, such that

$$0 < \eta_j < 1 - \frac{\alpha_j}{n}, \quad 1 + \log^+ t^{\frac{\alpha_j}{n}} \leq t^{\eta_j} \quad \text{if } t > C_0.$$

Denote $\eta = \sum_{j=1}^m \eta_j$, then by (4.7) if $|Q_{x_l}| > C_0$ we have

$$|Q_{x_l}|^{m - \frac{\alpha}{n} - \eta} \leq C \prod_{j=1}^m \int_{Q_{x_l}} \Phi^m(f_j). \tag{4.8}$$

Thus,

$$(m - \frac{\alpha}{n} - \eta) \log^+ (|Q_{x_l}|^{\frac{\alpha_j}{n}}) \leq C \frac{\alpha_j}{n} \log^+ \left(\prod_{j=1}^m \int_{Q_{x_l}} \Phi^m(f_j) \right).$$

By (4.7) again, we have

$$|Q_{x_l}|^{m - \frac{\alpha}{n}} \leq C \prod_{j=1}^m \left\{ 1 + \frac{\alpha_j}{n} \log^+ \left(\prod_{j=1}^m \int_{Q_{x_l}} \Phi^m(f_j) \right) \right\} \int_{Q_{x_l}} \Phi^m(f_j). \tag{4.9}$$

On the other hand, if $|Q_{x_l}| \leq C_0$, then it is easy to see $1 + \log^+ |Q_{x_l}|^{\frac{\alpha_j}{n}} \leq C$. Thus

$$|Q_{x_l}|^{m - \frac{\alpha}{n}} \leq C \prod_{j=1}^m \int_{Q_{x_l}} \Phi^m(f_j). \tag{4.10}$$

(4.9) and (4.10) yield that

$$|Q_{x_l}|^{m-\frac{\alpha}{n}} \leq C \prod_{j=1}^m \left\{ 1 + \frac{\alpha_j}{n} \log^+ \left(\prod_{j=1}^m \int_{Q_{x_l}} \Phi^m(f_j) \right) \right\} \int_{Q_{x_l}} \Phi^m(f_j). \quad (4.11)$$

Finally, by (4.4) and the definition of class $A_{((1, \dots, 1), \frac{n}{mn-\alpha})}$, we

$$\begin{aligned} \left(\int_{E_{1,k}} \nu_{\vec{\omega}}^{\frac{n}{mn-\alpha}} \right)^{\frac{mn-\alpha}{n}} &\leq \left(\sum_{h=1}^m \sum_{\sigma \in C_h^m} \sum_{l \in B_\sigma} \int_{Q_{x_l}} \nu_{\vec{\omega}}^{\frac{n}{mn-\alpha}} \right)^{\frac{mn-\alpha}{n}} \\ &\leq C \sum_{h=1}^m \sum_{\sigma \in C_h^m} \sum_{l \in B_\sigma} \left(\int_{Q_{x_l}} \nu_{\vec{\omega}}^{\frac{n}{mn-\alpha}} \right)^{\frac{mn-\alpha}{n}} \\ &\leq C \sum_{h=1}^m \sum_{\sigma \in C_h^m} \sum_{l \in B_\sigma} |Q_{x_l}|^{m-\frac{\alpha}{n}} \prod_{j=1}^m \inf w_j \\ &\leq C \sum_{h=1}^m \sum_{\sigma \in C_h^m} \sum_{l \in B_\sigma} \prod_{j=1}^m \left\{ 1 + \frac{\alpha_j}{n} \log^+ \left(\prod_{j=1}^m \int_{Q_{x_l}} \Phi^m(f_j) \right) \right\} \int_{Q_{x_l}} \Phi^m(f_j) w_j \\ &\leq C \sum_{h=1}^m \sum_{\sigma \in C_h^m} \sum_{l \in B_\sigma} \prod_{j=1}^m \left\{ 1 + \frac{\alpha_j}{n} \log^+ \left(\prod_{j=1}^m \int_{\mathbb{R}^n} \Phi^m(f_j) \right) \right\} \int_{Q_{x_l}} \Phi^m(f_j) w_j \\ &\leq C \prod_{j=1}^m \left\{ 1 + \frac{\alpha_j}{n} \log^+ \left(\prod_{j=1}^m \int_{\mathbb{R}^n} \Phi^m(f_j) \right) \right\} \int_{\mathbb{R}^n} \Phi^m(f_j) w_j. \end{aligned}$$

The proof of inequality (4.2) is finished.

Inequality (4.1) follows by taking $\alpha_j = \alpha/m < n$ in the above proof.

Proof of Theorem 1.4 and Corollary 1.1.

To prove Theorem 1.4, we follow the main steps as in [4], without changes till the last step by using the above Proposition 3.1, We will obtain Theorem 1.4.

To prove Corollary 1.1, similarly as in linear case [10], we define

$$\overline{I_{\alpha, \Pi b}}(\vec{f})(x) = \int_{(\mathbb{R}^n)^m} \frac{\prod_{j=1}^m |b_j(x) - b_j(y_j)|}{|(x - y_1, \dots, x - y_m)|^{mn-\alpha}} \prod_{i=1}^m |f_i(y_i)| d\vec{y}, \quad (4.12)$$

And careful check in the proof of Theorem 1.3-1.4 shows that Theorem 1.3-1.4 still hold for $\overline{I_{\alpha, \Pi b}}$. Note the fact that $\mathcal{M}_{\Pi b, \alpha}(\vec{f})(x) \leq \overline{I_{\alpha, \Pi b}}(|f_1|, \dots, |f_m|)(x)$, this implies Corollary 1.1.

Acknowledgements

The author want to express his sincerely thanks to the unknown referee for his or her valuable remarks and pointing out a problem in the previous version of this manuscript.

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